Journal of Statistical Physics, Vol. 51, Nos. 3/4, 1988

Noncommutative Cumulants for Stochastic Differential Equations and for Generalized Dyson Series

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Received December 24, 1987

For the stochastic equation $\dot{U} = VU$, Kubo's *ansätze* for $\langle U \rangle$ in the form of differential and integrodifferential equations is investigated and a new *ansatz* as an integral equation is added. Unique solutions in terms of noncommutative W- and K-cumulants are found by elementary functional differentiation, and expressions of van Kampen and Terwiel are recovered. For the cumulants we find simple recursion relations and prove the important cluster property. Surprisingly, it is found that the Gaussian approximation in the differential equation *ansatz* leads to positivity problems, while this is not the case with the integral and integrodifferential equation. The cumulant expansion technique is carried over to generalized Dyson series. In a companion paper we apply our results to quantum shot noise.

KEY WORDS: Stochastic differential equations; noncommutative cumulants; positivity problems.

1. INTRODUCTION AND MAIN RESULTS

Let $V(t)$, for real t, be a random matrix or operator, i.e., of the form $V(t; \omega)$, where ω ranges in a probability space over which one has to average. In many applications stochastic differential equations of the form

$$\frac{d}{dt} U(t, t_0) = V(t)\, U(t, t_0), \qquad U(t, t_0) = 1 \tag{1.1}$$

arise, and one is interested in the average $\langle U(t, t_0) \rangle$ and, in some cases, in its Fourier transform. Only rarely can this be calculated explicitly, and many approximation techniques have been devised (cf., e.g., Ref. 1). The

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0022-4715/88/0500-0691$06.00/0

tempting idea to use the Dyson series for $U(t, t_0)$ is in general not appropriate if one needs $\langle U(t, t_0)\rangle$ for large t, as in the Fourier transform, since there appear so-called secular terms growing like t^n if $\langle V(t_1)\cdots V(t_n)\rangle$ does not decay in the difference variables.

Kubo[2] advanced two interesting proposals, an *ansatz* for an integrodifferential equation and an *ansatz* for a differential equation for $\langle U(t, t_0)\rangle$; see *ansatz* (ID) and *ansatz* (D) below. Each *ansatz* leads to new types of cumulants and to an expansion of $\langle U(t, t_0)\rangle$ in terms of them. Kubo calculated the lowest cumulants for both cases.

Van Kampen[3] systematically investigated the differential equation for $\langle U(t, t_0)\rangle$ and found a solution for the cumulants to all orders with quite involved combinatorics; see also Fox.[4]

Terwiel[5] did not start from an *ansatz*, but used the projection operator technique of Mori[6] and Zwanzig[7] to construct directly an integrodifferential equation for $\langle U(t, 0)\rangle$. He also noted that his cumulants, which are different from those of van Kampen, satisfy the important cluster property to be discussed later. His derivation assumes that $\langle V(t)\rangle \equiv 0$.

In this paper we reconsider *ansatz* (ID) and *ansatz* (D) of Kubo, add an additional *ansatz* (I) in the form of an integral equation, and then extend them to generalized Dyson series. We clarify the conditions under which the respective kernels are uniquely determined by the *ansatz* and derive by elementary functional differentiations, without heavy combinatorics, compact expressions for them. From this, the original expressions of van Kampen and of Terwiel are recovered. If one has $\langle V(t)\rangle \neq 0$, *ansatz* (ID) and the expression of Terwiel have to be modified. Furthermore, we show that *ansatz* (I) and *ansatz* (ID) lead to the same cumulants and that the general expressions for the kernels are very simply related.

Thus we consider *ansätze* (I), (ID), and (D) given by

$$\text{(I)} \quad \langle U(t, t_0)\rangle = 1 + \int_{t_0}^{t} ds\, G_{\mathrm{I}}(t, s)\langle U(s, t_0)\rangle$$

$$\text{(ID)} \quad \frac{d}{dt}\langle U(t, t_0)\rangle = \int_{t_0}^{t} ds\, G_{\mathrm{ID}}(t, s)\langle U(s, t_0)\rangle$$

$$\text{(D)} \quad \frac{d}{dt}\langle U(t, t_0)\rangle = K_{\mathrm{D}}(t, t_0)\langle U(t, t_0)\rangle$$

It is clear that without further assumptions the kernels are not uniquely determined. For example, *ansatz* (D) may be integrated to give an integral equation of the form (I).

The main idea, implicit in the work of Kubo, is that (a) the kernels should be given by algebraic expressions in V which are independent of the particular form of V, i.e., by analytic functionals of $V(\cdot)$; (b) the kernels should depend only on values of $V(\cdot)$ for times in the intervals $[s, t]$ and $[t_0, t]$, respectively.

In Section 2 we show that under these two assumptions G_I is *uniquely* determined in terms of a specific type of noncommutative cumulants $\langle \cdot \rangle^W$. The G_I is given by a time-ordered exponential,

$$G_I(t, s) = \left\langle \mathscr{T} \left\{ \exp\left[\int_s^t V(t')\, dt' \right] \right\} V(s) \right\rangle^W$$

$$\equiv \sum_n \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{n-1}} dt_n \, \langle V(t_1) \cdots V(t_n)\, V(s) \rangle^W \tag{1.2}$$

With the abbreviation

$$\langle V(t_1) \cdots V(t_n) \rangle \equiv \langle 1 \cdots n \rangle \tag{1.3}$$

and similarly for $\langle \cdot \rangle^W$, the W-cumulants are given by $\langle 1 \rangle^W = \langle 1 \rangle$ and by the recursion relation

$$\langle 1 \cdots n \rangle = \langle 1 \cdots n \rangle^W + \sum_{i=n-1}^{1} \langle 1 \cdots i \rangle^W \langle i+1 \cdots n \rangle \tag{1.4}$$

The W stands for von Waldenfelds, in whose paper[8] we first met these cumulants explicitly, in a somewhat different context. For the lowest W-cumulants one easily finds from Eq. (1.4)

$$\langle 12 \rangle^W = \langle 12 \rangle - \langle 1 \rangle \langle 2 \rangle$$

$$\langle 123 \rangle^W = \langle 123 \rangle - \langle 12 \rangle \langle 3 \rangle - \langle 1 \rangle \langle 23 \rangle + \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle$$

$$\langle 1234 \rangle^W = \langle 1234 \rangle - \langle 123 \rangle \langle 4 \rangle - \langle 12 \rangle \langle 34 \rangle - \langle 1 \rangle \langle 234 \rangle$$
$$+ \langle 12 \rangle \langle 3 \rangle \langle 4 \rangle + \langle 1 \rangle \langle 23 \rangle \langle 4 \rangle + \langle 1 \rangle \langle 2 \rangle \langle 34 \rangle$$
$$- \langle 1 \rangle \langle 2 \rangle \langle 3 \rangle \langle 4 \rangle$$

A closed formula is given in Eq. (2.16). These W-cumulants differ from the ordinary ones even in the commutative case. It is seen that in their definition time-ordering is completely preserved.

Thus, $\langle U(t, t_0) \rangle$ obeys the integral equation

$$\langle U(t, t_0) \rangle = 1 + \int_{t_0}^t ds \left\langle \mathscr{T} \left\{ \exp\left[\int_s^t V(t')\, dt' \right] \right\} V(s) \right\rangle^W \langle U(s, t_0) \rangle \tag{1.5}$$

from which one obtains approximations by retaining only the first few terms for G_I in Eq. (1.2). In the stationary case Eq. (1.5) and its approximations are solved by Laplace transform [cf. Eq. (2.17)]. The time derivative of Eq. (1.5) is

$$\frac{d}{dt}\langle U(t,t_0)\rangle = \langle V(t)\rangle\langle U(t,t_0)\rangle + \int_{t_0}^{t} ds \left\langle V(t)\mathscr{T}\left\{\exp\left[\int_s^t V(t')\,dt'\right]\right\} V(s)\right\rangle^{W} \langle U(s,t_0)\rangle \tag{1.6}$$

which is an integrodifferential equation for $\langle U(t,t_0)\rangle$. For $\langle V(t)\rangle \equiv 0$ the first term is absent, and the equation becomes a compact form of Terwiel's expression. For $\langle V\rangle \neq 0$ the *ansatz* (ID) in its above form clearly has no solution, since for $t = t_0$ the rhs vanishes, while the lhs equals $\langle V(t_0)\rangle$. We show in Section 2 that Eq. (1.6) is the unique solution for its extension to the case $\langle V\rangle \neq 0$.

In Section 3 we give a short proof by elementary functional differentiation for the unique form of K_D in terms of K-cumulants $\langle\cdot\rangle^K$ (K after van Kampen). *Ansatz* (D) becomes

$$\frac{d}{dt}\langle U(t,t_0)\rangle = \left\langle V(t)\mathscr{T}\exp\left[\int_{t_0}^{t} V(t')\,dt'\right]\right\rangle^{K} \langle U(t,t_0)\rangle \tag{1.7}$$

where the expression for K_D is to be understood similarly as in Eq. (1.2). The K-cumulants are given by $\langle 1\rangle^K = \langle 1\rangle$ and by the recursion relation

$$\langle 1\cdots n\rangle = \langle 1\cdots n\rangle^K + \sum_{\{i_1<\cdots<i_r\}\cup\{j_1<\cdots<j_s\}=\{2,\ldots n\}} \langle 1 i_1\cdots i_r\rangle^K \langle j_1\cdots j_s\rangle \tag{1.8}$$

where $r+s=n-1$ and $r\geqslant 0$, $s\geqslant 1$. This relation seems to be new for the noncommutative case; for commuting expectations see Ref. 9. One easily calculates from Eq. (1.8)

$$\begin{aligned}\langle 12\rangle^K &= \langle 12\rangle - \langle 1\rangle\langle 2\rangle = \langle 12\rangle^W \\ \langle 123\rangle^K &= \langle 123\rangle - \langle 12\rangle\langle 3\rangle - \langle 13\rangle\langle 2\rangle - \langle 1\rangle\langle 23\rangle \\ &\quad + \langle 1\rangle\langle 2\rangle\langle 3\rangle + \langle 1\rangle\langle 3\rangle\langle 2\rangle\end{aligned} \tag{1.9}$$

In Section 3 a closed formula for $\langle 1\cdots n\rangle^K$ is given, which is equivalent to the prescription of van Kampen.[3] With this identification the differential

equation for $\langle U \rangle$, Eq. (1.7), becomes a compact form of van Kampen's result. Another type of cumulant closely related to K-cumulants was used by Roger.[10]

In Section 3 we prove the cluster property for K-cumulants, which was not mentioned in Ref. 3. The cluster property means that if $V(t_1),..., V(t_n)$ splits into two statistically independent clusters $V(t_1),..., V(t_m)$ and $V(t_{m+1}),..., V(t_n)$ so that $\langle V(t_1) \cdots V(t_n) \rangle$ factorizes, then the cumulant vanishes.

The cluster property is crucial for approximations to the formally exact equations (1.5)–(1.7). In many applications this statistical independence holds at least asymptotically for sufficiently large time separation of the clusters, and so the cumulants will go to zero in the difference variables. If this decrease is sufficiently rapid, one can hope to get a good approximation of the time-ordered exponential with the first few cumulants. In first-order approximation the integral and differential equations give the same result. The second-order approximation will be called W- and K-Gaussian approximation, respectively. In this context certain nonpositivity problems arise already for the K-Gaussian approximation, to be discussed in Section 4.

In Section 5 we show that the above procedures of cumulant expansions can be carried over from stochastic differential equations to expressions given by generalized Dyson series; cf. also Ref. 11. To motivate this, we note that the solution $U(t, t_0)$ of Eq. (1.1) can be written as a Dyson series, and hence

$$\langle U(t, t_0) \rangle = \mathbf{1} + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \, \langle V(t_1) \cdots V(t_n) \rangle \tag{1.10}$$

In applications one sometimes meets generalized Dyson series of the form

$$F(t, t_0) = \mathbf{1} + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t}^{t_{n-1}} dt_n \, f_n(t_1,..., t_n) \tag{1.11}$$

where the f_n are matrix- or operator-valued functions, which may arise from averages of nonproduct expressions. Abbreviating

$$f_n(t_1,..., t_n) \equiv \langle 1 \cdots n \rangle$$

one can define W- and K-cumulants

$$f_n^{\mathrm{W}}(t_1,..., t_n), \qquad f_n^{\mathrm{K}}(t_1,..., t_n)$$

in exactly the same way as before by the recursion relations (1.1) and (1.8); for example,

$$f_2^{\mathrm{W}}(t_1, t_2) = f_2(t_1, t_2) - f_1(t_1) f(t_2) = f_2^{\mathrm{K}}(t_1, t_2)$$

$$f_3^{\mathrm{W}}(t_1, t_2, t_3) = f_3(t_1, t_2, t_3) - f_1(t_1) f_2(t_2, t_3) - f_2(t_1, t_2) f_1(t_3) + f_1(t_1) f_1(t_2) f_1(t_3)$$

and so on. One has again the cluster property, i.e., if f_n factorizes, e.g., due to independence of two clusters, then f_n^{W} and f_n^{K} vanish.

The integral equation (1.5) generalizes to

$$F(t, t_0) = \mathbf{1} + \int_{t_0}^{t} ds\, G_{\mathrm{Dy}}(t, s)\, F(s, t_0) \tag{1.12}$$

with

$$G_{\mathrm{Dy}}(t, s) = f_1(s) + \sum_1^\infty \int_s^t dt_1 \cdots \int_s^{t_{n-1}} dt_n\, f_{n+1}^{\mathrm{W}}(t_1, \ldots, t_n, s)$$

$$\equiv -\frac{\partial}{\partial s} F^{\mathrm{W}}(t, s) \tag{1.13}$$

The differential equation carries over as

$$\frac{d}{dt} F(t, t_0) = K_{\mathrm{Dy}}(t, t_0)\, F(t, t_0) \tag{1.14}$$

with

$$K_{\mathrm{Dy}}(t, t_0) = f_1(t) + \sum_1^\infty \int_{t_0}^t dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n\, f_{n+1}^{\mathrm{K}}(t, t_1, \ldots, t_n)$$

$$\equiv \frac{\partial}{\partial t} F^{\mathrm{K}}(t, t_0) \tag{1.15}$$

Again the hope is that, due to the cluster property, the cumulant expansions for G_{Dy} and K_{Dy} converge much more rapidly than the original series in Eq. (1.11) and that it is sufficient to retain only the first few terms.

In the Appendix we show how the cumulant expansions fit into a more general nonprobabilistic framework and indicate algebraic and power series techniques to deal efficiently with the combinatorics.

In a companion paper we apply our present results to quantum shot noise. In this case the noncommutative cumulant expansions turn out to allow partial summations, which result in expansions in powers of the pulse density v. Also in preparation are applications to the theory of spectral-line broadening.

2. THE INTEGRAL EQUATION FOR ⟨U⟩ AND W-CUMULANTS

2.1. Derivation of the integral equation

The stochastic differential equation (1.1) resembles a Schrödinger equation, with a possible i absorbed in V. We will call $V(t)$ the "potential," By assumption (a), *ansatz* (I) holds for all potentials, in particular also for

$$V_h(t) := h(t)\, V(t)$$

where $h(t)$ is a scalar function. By assumption (a), the kernel G_{I} must be a formally analytic expression in $h(\,\cdot\,)$, where, by assumption (b), only values of h for times between s and t enter. Furthermore, because of the factor ds, it follows for dimensional reasons that G_{I} must contain an explicit factor $h(s)$.[3]

Hence G_{I} has the general form

$$G_{\mathrm{I}}(hV; t, s) = h(s) \sum_{n=0}^{\infty} \int_s^t d\tau_1 \cdots \int_s^t d\tau_n \times \kappa_n(V; t, t_1, \ldots, t_n, s)\, h(t_1) \cdots h(t_n) \tag{2.1}$$

We note that powers of $h(t_i)$ cannot appear, because such powers appear neither in $\langle U(t, t_0)\rangle$ nor in $\langle U(s, t_0)\rangle$. Symmetrizing κ_n in $t_1, \ldots, t_n$, we can write the rhs of Eq. (2.1) as a time-ordered integral in the form

$$G_{\mathrm{I}}(hV; t, s) = \left[\sum_{n=0}^{\infty} \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{n-1}} dt_n \times k_n(V; t, t_1, \ldots, t_n, s)\, h(t_1) \cdots h(t_n) \right] h(s) \tag{2.2}$$

[3] This also results from a simple scaling argument by scaling the time according to $dt' = h(t)\, dt$ and considering the stochastic differential equation and *ansatz* (I) with a t' derivative.

Now we employ straightforward functional derivatives. Let $t > s_1 > \cdots > s_n > t_0$. We consider *ansatz* (I) with potential hV and apply[4]

$$\frac{\delta}{\delta h(s_1)} \cdots \frac{\delta}{\delta h(s_n)}\bigg|_{h=0} \tag{2.3}$$

to both sides. Now the crucial point is that because of the appearance of the explicit factor $h(s)$ in G_I, one of the functional derivatives, with s_i say, has to hit $h(s)$, producing $\delta(s-s_i)$, since otherwise the contribution would vanish after setting $h=0$. Then, because of the time ordering, the derivatives with times earlier than s_i apply to $\langle U(s, t_0)\rangle$ only and the later ones to G_I only. The δ-function eliminates the integral over s, and with the Dyson series for $\langle U(t', t_0)\rangle$ and with Eq. (2.2) we obtain

$$\begin{aligned}\langle V(s_1)\cdots V(s_n)\rangle = {} & k_{n-1}(V; t, s_1,..., s_n) \\ & + k_{n-2}(V; t, s_1,..., s_{n-1})\langle V(s_n)\rangle + \cdots \\ & + k_0(V; t, s_1)\langle V(s_2)\cdots V(s_n)\rangle\end{aligned} \tag{2.4}$$

This recursive relation for k_n yields

$$\begin{aligned} k_0(V; t, s_1) &= \langle V(s_1)\rangle \\ k_1(V; t, s_1, s_2) &= \langle V(s_1)\, V(s_2)\rangle - \langle V(s_1)\rangle\langle V(s_2)\rangle \end{aligned}$$

which is $\langle V(s_1)\, V(s_2)\rangle^{\mathrm{W}}$. Indeed, the recursion relation (2.4) is the same as that in Eq. (1.4), and it is seen immediately that $k_{n-1}(V; t, s_1,..., s_n)$ is uniquely determined, does not depend on t, and is a *multilinear* function of $V(s_1),..., V(s_n)$. We may therefore put

$$\langle V(s_1)\cdots V(s_n)\rangle^{\mathrm{W}} := k_{n-1}(V; t, s_1,..., s_n) \tag{2.5}$$

[4] The only fact needed is the formula

$$\frac{\delta}{\delta h(t')} h(t) = \delta(t-t')$$

Then all functional derivatives can be pulled under the integral signs, and the usual differentiation rules hold. In particular, for $t > s_1 > \cdots > s_m > t_0$ one has

$$\begin{aligned}&\frac{\delta}{\delta h(s_1)} \cdots \frac{\delta}{\delta h(s_m)}\bigg|_{h=0} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, f(t_1,..., t_n)\, h(t_1)\cdots h(t_n) \\ &\quad = \delta_{nm} f(s_1,..., s_m)\end{aligned}$$

and Eq. (2.4) becomes Eq. (1.4). Inserting Eq. (2.5) into Eq. (2.2), we obtain for $h \equiv 1$, using the multilinearity of $\langle \cdot \rangle^{\mathrm{W}}$,

$$G_{\mathrm{I}}(V; t, s) = \sum_{n=0}^{\infty} \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{n-1}} dt_n \times \langle V(t_1) \cdots V(t_n)\, V(s) \rangle^{\mathrm{W}} \tag{2.6}$$

$$= \left\langle \mathscr{T} \left\{ \exp\left[\int_s^t V(t')\, dt' \right] \right\} V(s) \right\rangle^{\mathrm{W}} = -\frac{\delta}{\delta s} \left\langle \mathscr{T} \exp\left[\int_s^t V(t')\, dt' \right] \right\rangle^{\mathrm{W}} \tag{2.7}$$

Hence, *if* $\langle U(t, t_0) \rangle$ satisfies an integral equation as in *ansatz* (I), the kernel is given by Eq. (2.7). To show the converse, that the integral equation is indeed satisfied, we give two alternative arguments. The pedestrian way is to integrate Eq. (1.4) over $t > t_1 > \cdots > t_n > t_0$ and to sum over n. The lhs then becomes $\langle U(t, t_0) \rangle$, and on the rhs one can regroup terms and make a change of integration variables to obtain in Eq. (1.5).

A more elegant way is as follows. We replace V by hV, use Eq. (2.6) for $G_{\mathrm{I}}(hV; t, s)$, and insert the Dyson series (1.10) for $\langle U(s, t_0) \rangle$. The rhs of Eq. (1.5) then becomes a sum over products of two time-ordered integrals. Taking care of the integration domains by step functions, these integrals can be written as multiple integrals from t_0 to t. Hence the rhs of Eq. (1.5) can be written, with hV replacing V, in the form

$$+ \sum_{m=1}^{\infty} \int_{t_0}^t dt_1 \int_{t_0}^t dt_2 \cdots \int_{t_0}^t dt_m\, q_m(V; t_1, \ldots, t_m)\, h(t_1) \cdots h(t_m) \tag{2.8}$$

Symmetrizing q_m in $t_1, \ldots, t_m$, we can write this with time-ordered integrals as

$$1 + \sum_{m=1}^{\infty} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{m-1}} dt_m\, m!\, q_m^{\mathrm{S}}(V; t_1, \ldots, t_m)\, h(t_1) \cdots h(t_m) \tag{2.9}$$

The functional derivatives of this at $h = 0$ yield $m!\, q_m^{\mathrm{S}}$, $m = 1, 2, \ldots$, which therefore coincide with the functional derivatives of the original expression, i.e.,

$$m!\, q_m^{\mathrm{S}} = \langle V(t_1) \cdots V(t_n) \rangle^{\mathrm{W}} + \sum_j \langle V(t_1) \cdots V(t_j) \rangle^{\mathrm{W}} \langle V(t_{j+1}) \cdots \rangle = \langle V(t_1) \cdots V(t_n) \rangle \tag{2.10}$$

where the second equality comes from Eq. (1.4). Thus, the expression in Eq. (2.9), with $h \equiv 1$, equals $\langle U(t, t_0) \rangle$.

2.2. Cluster Property

This property is already implied by the integral equation. Indeed, if the values of the potentials for times smaller than some t_c are independent of those for times greater than t_c, then one has for $t \geqslant s > t_c > t_0$

$$\langle U(s, t_0)\rangle = \langle U(s, t_c)\rangle \langle U(t_c, t_0)\rangle \tag{2.11}$$

Splitting the ds integral in Eq. (1.2) into one from t_0 to t_c and one from t_0 to t and inserting Eq. (2.11), one immediately finds

$$\langle U(t_c, t_0)\rangle = 1 + \int_{t_0}^{t_c} ds \left\langle \mathscr{T}\left\{\exp\left[\int_s^t dt'\, V(t')\right]\right\} V(s)\right\rangle^{\mathrm{W}} \langle U(s, t_0)\rangle \tag{2.12}$$

Since this also holds with V replaced by hV, identically in h, the kernel cannot depend on hV values for times larger than t_c. Hence

$$\langle V(t_1)\cdots V(t_n)\rangle^{\mathrm{W}} = 0 \qquad \text{if} \quad t_1 > t_c > t_n \tag{2.13}$$

The cluster property also follows easily from the recursion relation (1.4) by induction on n as follows. If $V(t_1),\ldots, V(t_m)$ and $V(t_{m+1}),\ldots, V(t_n)$ are independent, then

$$\begin{aligned}\langle 1\cdots m\rangle\langle m+1\cdots n\rangle &= \langle 1\cdots n\rangle \\ &= \langle 1\cdots n\rangle^{\mathrm{W}} + \sum_{i=n-1}^{1} \langle 1\cdots i\rangle^{\mathrm{W}} \langle i+1\cdots n\rangle\end{aligned} \tag{2.14}$$

For $n=2$, $\langle V(t_1)\, V(t_2)\rangle^{\mathrm{W}} = 0$ if one has two independent clusters. We assume this to hold for $n-1$. Then the sum in Eq. (2.14) is only from m to 1, and $\langle i+1\cdots n\rangle$ factorizes. Hence, the rhs becomes

$$\langle 1\cdots n\rangle^{\mathrm{W}} + \langle 1\cdots m\rangle\langle m+1\cdots n\rangle$$

which implies $\langle 1\cdots n\rangle^{\mathrm{W}} = 0$.

2.3. Formula for W-Cumulants

It may seem from Eq. (1.4) or (2.4) and (2.5) that W-cumulants are defined for $t_1 > \cdots > t_n$ only. This is not so. Any sequence A_1, A_2,... of random operators or matrices can be considered to define a piecewise constant potential, and so $\langle A_1\cdots A_n\rangle^{\mathrm{W}}$ may be defined by Eq. (1.4). To give a

closed expression for $\langle\cdot\rangle^{\mathrm{W}}$, we consider "intervals" I of integers, $I=\{l, l+1,..., r\}$, and put

$$\langle A_l A_{l+1}\cdots A_r\rangle =: \langle I\rangle \tag{2.15}$$

Similarly for $\langle I\rangle^{\mathrm{W}}$. Furthermore, $I_1\circ I_2$ denotes union of *adjoining* intervals, i.e.,

$$I_1=\{l_1,..., r_1\},\qquad I_2=\{r_1+1,..., r_2\}$$
$$I_1\circ I_2=\{l_1,..., r_2\}$$

Then one has[8,12]

$$\langle I\rangle^{\mathrm{W}}=\sum_{I_1\circ\cdots\circ I_K=I}(-1)^{K-1}\langle I_1\rangle\cdots\langle I_K\rangle \tag{2.16}$$

where the sum is over all partitions of I into adjoining nonempty intervals. This is easily proved directly by induction; it also follows from Eq. (A.19).

It is apparent from the rhs of Eq. (2.16) that the ordering is completely preserved.

2.4. Stationarity of W-Cumulants

If the process $V(t)$ is stationary, then it is seen from Eq. (2.16) that its W-cumulants depend on time differences only. The same then holds for the kernel G_{I}, so that the integral equation (1.5) as well as its approximations can be solved by *Laplace transform*,

$$\mathscr{L}_p\{\langle U(t,0)\rangle\}=(p-p\mathscr{L}_p\{G\})^{-1} \tag{2.17}$$

2.5. The Integrodifferential Equation

It is seen for $t=t_0$ that *ansatz* (ID) can only hold for $\langle V(t_0)\rangle=0$. Assuming, therefore, first that $\langle V(\cdot)\rangle\equiv 0$, one easily shows by the previous techniques that G_{ID} is uniquely determined by assumptions (a) and (b) and that one obtains Eq. (1.6) with the first term on the rhs omitted. Due to noncommutativity, subtracting or adding the expectation value $\langle V(\cdot)\rangle$ is not quite so trivial as in the classical commutative case. Here one has to consider the analog of the quantum mechanical interaction picture. Doing this and using the cluster property, one arrives after some calculation at the full equation (1.6).

One can of course also show the validity of Eq. (1.6) directly and very easily by the same arguments as in Eqs. (2.8)–(2.10).

Remark. We note that for the uniqueness of G_I and G_{ID} only part of assumption (a) of Section 1 was needed, since only the potentials $V_h(t) = h(t)\, V(t)$ entered, with $V(t)$ fixed. Also, the expectation or average $\langle \cdot \rangle$ was fixed in the discussion.

3. K-CUMULANTS AND THE DIFFERENTIAL EQUATION FOR $\langle U \rangle$

3.1. Derivation of the differentiation equation

Finding the kernel K_D for *ansatz* (D) by functional differentiation is even simpler than for the integral equation. Again we replace V by hV. Then on the lhs of *ansatz* (D), $h(t)$ appears as a factor and hence it appears also as a factor on the rhs. We proceed as in the step from Eq. (2.1) to Eq. (2.2) to go from a multiple time integral to a time-ordered integral, by symmetrization, and obtain from *ansatz* (D) an equation of the form

$$\begin{aligned} &h(t)\langle V(t)\, U(hV; t, t_0)\rangle \\ &\quad = h(t)\Bigg[\sum_{n=0}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \\ &\qquad \times \hat{k}_n(t, t_1, \ldots, t_n, t_0)\, h(t_1) \cdots h(t_n)\Bigg] \langle U(hV; t, t_0)\rangle \end{aligned} \tag{3.1}$$

We divide by $h(t)$, put $t \equiv s_1$, and apply, for $t \equiv s_1 > s_2 > \cdots > s_n > t_0$, $n \geqslant 1$,

$$\frac{\delta}{\delta h(s_2)} \cdots \frac{\delta}{\delta h(s_n)}\bigg|_{h=0}$$

As opposed to Eq. (2.2), there is now no time ordering between the two factors on the rhs of Eq. (3.1). Therefore, the *Leibnitz* rule for multiple derivatives of a product gives

$$\begin{aligned} &\langle V(s_1) \cdots V(s_n)\rangle \\ &\quad = \hat{k}_{n-1}(s_1, \ldots, s_n, t_0) \\ &\qquad + \sum_{\{i_1 < \cdots < i_r\} \cup \{j_1 < \cdots < j_s\} = \{2, \ldots, n\}} \hat{k}_{n-r}(s_1, s_{i_1}, \ldots, s_{i_r}) \\ &\qquad \times \langle V(s_{j_1}) \cdots V(s_{j_s})\rangle \end{aligned} \tag{3.2}$$

where $r+s=n-1$ and $r \geqslant 0$, $s \geqslant 1$. The symbol $\cup$ denotes union of disjoint sets. Disjointness is automatically fulfilled due to the conditions on r and s. This recursive relation determines $\hat{k}_n$ uniquely, and it is a *multilinear* function in the V's, independent of t_0, so that we may define

$$\langle V(s_1) \cdots V(s_n) \rangle^{\mathrm{K}} := \hat{k}_{n-1}(s_1, \ldots, s_n, t_0) \tag{3.3}$$

Then Eq. (3.2) is identical to Eq. (1.8). Inserting Eq. (3.3) into Eq. (3.1) and using the multilinearity of $\langle \cdot \rangle^{\mathrm{K}}$, we obtain for $h \equiv 1$

$$\begin{aligned} K_{\mathrm{D}}(V; t, t_0) &= \sum_{n=0}^{\infty} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \\ &\quad \times \langle V(t)\, V(t_1) \cdots V(t_n) \rangle^{\mathrm{K}} \\ &= \left\langle V(t) \mathscr{T} \exp\left[\int_{t_0}^{t} dt'\, V(t') \right] \right\rangle^{\mathrm{K}} \end{aligned} \tag{3.4}$$

Conversely, with the same procedure of functional differentiation as in Eqs. (2.8)–(2.10), it is immediately seen that *ansatz* (D) is indeed fulfilled for this form of K_{D}. The pedestrian way of integrating the recursion relation (1.8) or (3.2) in a time-ordered way, i.e., over $t > s_2 > \cdots > s_n > t_0$, and summing over n becomes very tedious in this case.

3.2. Cluster Property

As for W-cumulants, the cluster property for $\langle \cdot \rangle^{\mathrm{K}}$ is already implied by the differential equations. To see this, let the values of the potential for times above t_c be independent of those for times below t_c. Then, for $t > t_c > t_0$,

$$\langle V(t)\, U(t, t_0) \rangle = \langle V(t)\, U(t, t_c) \rangle \langle U(t_c, t_0) \rangle$$

From this one obtains with Eq. (1.7)

$$\langle V(t)\, U(t, t_c) \rangle = \left\langle V(t) \mathscr{T} \exp\left[\int_{t_0}^{t} dt'\, V(t') \right] \right\rangle^{\mathrm{K}} \langle U(t, t_c) \rangle \tag{3.5}$$

Since this also holds with V replaced by hV, identically in h, the kernel cannot depend on hV values for times smaller than t_c. Hence

$$\langle V(t_1) \cdots V(t_n) \rangle^{\mathrm{K}} = 0 \qquad \text{if} \quad t_1 > t_c > t_n \tag{3.6}$$

One can also prove the cluster property with the recursion relation (1.8) and induction, as in Eq. (2.14).

3.3. Formulas for K-Cumulants

Let V_i, $i = 1, 2, \ldots$, be random operators or matrices. Then, by a simple argument (cf. Appendix), one has in the sense of formal power series

$$\langle V_1 \cdots V_n \rangle^{\mathrm{K}} = \frac{\delta}{\delta x_2} \cdots \frac{\delta}{\delta x_n}\bigg|_{\mathbf{x}=0} \times \left(\langle V_1 e^{x_2 V_2} \cdots e^{x_n V_n} \rangle \langle e^{x_2 V_2} \cdots e^{x_n V_n} \rangle^{-1} \right) \tag{3.7}$$

In the Appendix we use this to discuss the relationship between K-cumulants and ordinary (commutative) cumulants.

For higher n the evaluation of Eq. (3.7) becomes tedious. For an alternative expression we consider sets of integers $\Lambda = \{\lambda_1 < \lambda_2 < \cdots < \lambda_r\}$ and introduce, as in Eq. (2.15), the abbreviations

$$\langle \Lambda \rangle := \langle V_{\lambda_1} \cdots V_{\lambda_r} \rangle, \qquad \langle \Lambda \rangle^{\mathrm{K}} := \langle V_{\lambda_1} \cdots V_{\lambda_r} \rangle^{\mathrm{K}} \tag{3.8}$$

In the Appendix we show more elegantly, without Eq. (3.7),

$$\langle \Lambda \rangle^{\mathrm{K}} = \sum_{\substack{\Lambda_1 \cup \cdots \cup \Lambda_l = \Lambda \\ \lambda_1 \in \Lambda_1, \Lambda_i \neq \varnothing}} (-1)^{l-1} \langle \Lambda_1 \rangle \cdots \langle \Lambda_l \rangle \tag{3.9}$$

The summation is over all tuples $\{\Lambda_1, \ldots, \Lambda_n\}$ arising from partitions of Λ into disjoint subsets, with $\lambda_1 \in \Lambda_1$. Note that, for example, with $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ also $\{\Lambda_1, \Lambda_3, \Lambda_2\}$ appears in the sum. Proof is by induction or as in the Appendix. A close look at the paper of van Kampen[3] shows that Eq. (3.9) agrees with his prescription for the cumulant construction.

In contrast to the W-cumulants, the K-cumulants preserve time ordering only in a restricted sense. Within each $\langle \Lambda_i \rangle$ the operators are ordered, but, for example, $\langle \Lambda_l \rangle$ may contain V_2. This break of time order is related to the nonpositivity aspect discussed in the next section.

Stationary. If the process $V(t)$ is stationary, then, by Eq. (3.9), the K-cumulants depend on time differences only.

4. POSITIVITY PROBLEMS FOR THE GAUSSIAN APPROXIMATION

In this section we assume $\langle V(t) \rangle = 0$. Then the simplest approximation in the integral and differential equations is to retain only

the second cumulants in the expansion of the kernels so that, in second order in V,

$$G_{\mathrm{I}}^{(2)}(t, s) = \int_s^t dt_1 \, \langle V(t_1) \, V(s) \rangle$$
$$K_{\mathrm{D}}^{(2)}(t, t_0) = \int_{t_0}^t dt_1 \, \langle V(t) \, V(t_1) \rangle \tag{4.1}$$

For $\langle U \rangle$ one obtains two in general different approximations, one from

$$\langle U(t, t_0) \rangle_{\mathrm{I}}^{(2)} = 1 + \int_{t_0}^t ds \, G_{\mathrm{I}}^{(2)}(t, s) \langle U(t, t_0) \rangle_{\mathrm{I}}^{(2)} \tag{4.2}$$

and the other from

$$\frac{d}{dt} \langle U(t, t_0) \rangle_{\mathrm{D}}^{(2)} = K_{\mathrm{D}}^{(2)}(t, t_0) \langle U(t, t_0) \rangle_{\mathrm{D}}^{(2)} \tag{4.3}$$

We will call these the W-Gaussian and K-Gaussian approximations, respectively.

In the c-number commutative case it is well known that, due to the Marcinkiewicz theorem,[13,9] one obtains negative parts in the supposedly positive measure if one terminates the cumulants at some $N \geqslant 3$, while the Gaussian approximation is acceptable in this respect.

In the noncommutative case, however, already the K-Gaussian approximation leads to a related positivity problem. An elementary way to see this is as follows. Let V be skew-Hermitian, so that $U(t, t_0)$ is unitary. Then

$$\sum \lambda_i \bar{\lambda}_j \langle U(t_i, t_j) \rangle = \left\langle \left\{ \sum \lambda_i U(t_i, 0) \right\} \left\{ \sum \lambda_j U(t_j, 0) \right\}^* \right\rangle \geqslant 0 \tag{4.4}$$

In the stationary case one has

$$\langle U(t_i, t_j) \rangle = \langle U(t_i - t_j, 0) \rangle$$

so that $\langle U(t, 0) \rangle$ is a positive-definite function, or equivalently, its Fourier transform is positive. This positivity is violated by the K-Gaussian approximation in the noncommutative case, as indicated by the following argument. The $2n$th moment of the Fourier transform is positive and equals, up to a sign, the $2n$th derivative of $\langle U(t, 0) \rangle$ at $t = 0$. Hence

$$(-1)^n \left. \frac{d^{2n}}{dt^{2n}} \right|_{t=0} \langle U(t, 0) \rangle \geqslant 0$$

We denote $\Phi^{(n)} := \langle V^{(n)}(0)\, V(0)\rangle$ and calculate

$$(-1)^3 \frac{d^6}{dt^6} \langle U(t,0)\rangle_{\mathrm{D}}^{(2)} = -(\Phi^{(4)} + 10\Phi''\Phi + 10\Phi'^2 + 5\Phi\Phi'' + 15\Phi^3) \quad (4.5)$$

Now, it is easy to see by stationarity that $\Phi^{(4)}$, Φ'', and Φ are Hermitian and that Φ' is skew-Hermitian. Hence, if Φ and Φ'' do not commute, then the expression in Eq. (4.5) is not Hermitian and, *a fortiori*, not positive. It is not difficult to construct a process where this happens.

For the W-Gaussian approximation this positivity problem does not occur. One can show that it always satisfies the positivity condition (4.5) for unitary U. Of course, if the K-Gaussian approximation is a good approximation, the positivity violation will in general be minor.

The deeper reason for this qualitative difference is that the n-point correlation functions reconstructed via the W-recursion relation (1.4) from the second cumulants satisfy the usual positivity condition for moments, while in the noncommutative case the n-point correlation functions reconstructed via the K-recursion relation from the second cumulants do *not* satisfy this moment positivity condition. We briefly touch on this question in the Appendix.

5. THE CUMULANT EXPANSIONS FOR GENERALIZED DYSON SERIES

The restriction to time-ordered integration in Eq. (1.11) is no loss of generality, since any expression of the form

$$F(t, t_0) = \mathbf{1} + \sum_1^\infty \int_{t_0}^t dt_1 \cdots \int_{t_0}^t dt_n\, q_n(t_1, \ldots, t_n) \quad (5.1)$$

can be written as a generalized Dyson series with f_n the symmetrization of q_n, just as in Eqs. (2.8), (2.9).

We first consider the integral equation (1.12). Similar to Section 2, let $F(h; t, t_0)$ be given by Eq. (1.11) with f_n replaced by $f_n h(t_1)\cdots h(t_n)$, and analogously for $G_{\mathrm{Dy}}(h; t, s)$. By the same reasoning as in Section 2, G_{Dy} contains the explicit factor $h(s)$ and has the same general form as in Eq. (2.2). Functional differentiation gives the analog of Eq. (2.4) with $\langle V(s_i)\cdots V(s_n)\rangle$ replaced by $f_{n-i+1}(s_i, \ldots, s_n)$. This gives the recursion relation for f_n^{W}, and we obtain Eq. (1.13) for G_{Dy}. Sufficiency follows as in Eqs. (2.8), and (2.9), and the same remark on uniqueness as at the end of Section 2 about the partial use of assumption (a) applies.

In an analogous way one obtains the differential equation for $F(t, t_0)$, Eqs. (1.14) and (1.15), and the uniqueness of its kernel. For a somewhat more involved derivation of Eqs. (1.13) and (1.15) see Ref. 11.

APPENDIX. CUMULANTS FOR NONCOMMUTATIVE STATES AS POWER SERIES

A1. Introduction

The conventional probabilistic context for cumulants is unnecessarily restrictive. The proper framework[14] is to consider a set $\mathcal{M}$ of (abstract, noncommuting) elements a_i, which replace $V(t_i)$. Polynomials in the a_i as indeterminants form an algebra $\mathcal{M}$ (the free algebra generated by $\mathcal{M}$). We assume an involution, $a \to a^*$, in $\mathcal{M}$ which is extended to $\mathcal{M}$ to make it a *-algebra.

The expectation or average is replaced by an operator-valued state on $\mathcal{M}$, denoted interchangeably by $m(\cdot)$ or $\langle \cdot \rangle$. A state is a linear functional on $\mathcal{M}$ that is positive on positive elements in $\mathcal{M}$ and normalized, $m(1) = 1$. The values of m may also lie in some abstract *-algebra, which will be assumed fixed in the following.

Let $a_1, a_2, \ldots, \varepsilon \mathcal{M}$. For $\Lambda = \{\lambda_1 < \cdots < \lambda_k\}$ we put $a_\Lambda := a_{\lambda_1} \cdots a_{\lambda_k}$ and $a_\varnothing = 1$. For linear functionals S and T on $\mathcal{M}$ we define their convolution as in Ref. 14 by

$$S * T(a_\Lambda) = \sum_{\Lambda_1 \cup \Lambda_2 = \Lambda} S(a_{\Lambda_1})\, T(a_{\Lambda_2}) \tag{A.1}$$

where Λ_i may be empty and where in the sum both $\{\Lambda_1, \Lambda_2\}$ and $\{\Lambda_2, \Lambda_1\}$ appear. This defines an associative, in general noncommutative, product. As in Ref. 14 for commutative states one has in the sense of formal power series

$$S * T(e^{x_1 a_1} \cdots e^{x_n a_n}) = S(e^{x_1 a_1} \cdots e^{x_n a_n})\, T(e^{x_1 a_1} \cdots e^{x_n a_n}) \tag{A.2}$$

If $f(z) = \sum \rho_n z^n$ is an analytic function, one can define, with $T_*^n := T * \cdots * T$,

$$f_*(T) := \sum \rho_n T_*^n$$

Theorem 2.1 of Ref. 14 also holds here,

$$f_*(T)(e^{x_1 a_1} \cdots e^{x_n a_n}) = f(T(e^{x_1 a_1} \cdots e^{x_n a_n})) \tag{A.3}$$

For convergence properties see Corollaries 2.3 and 2.7 of Ref. 14.

For a scalar- (or commutative-) valued state m the cumulant functional m^c is defined in Ref. 14 by

$$m^c := \log_* m, \qquad m := \exp_* m^c \tag{A.4}$$

Equation (A.3) gives

$$\langle \exp(x_1 a_1) \cdots \exp(x_n a_n) \rangle^c = \log \langle \exp(x_1 a_1) \cdots \exp(x_n a_n) \rangle \tag{A.5}$$

$$\langle \exp(x_1 a_1) \cdots \exp(x_n a_n) \rangle = \exp \langle \exp(x_1 a_1) \cdots \exp(x_n a_n) \rangle^c \tag{A.6}$$

and one obtains $\langle \cdot \rangle$ and $\langle \cdot \rangle^c$ by differentiating at $x = 0$. Applying $\partial/\partial x_1$ at $x_1 = 0$, one gets from Eq. (A.6), for a *commutative* state,

$$\langle a_1 e^{x_2 a_2} \cdots e^{x_n a_n} \rangle = \langle a_1 e^{x_2 a_2} \cdots e^{x_n a_n} \rangle^c \langle e^{x_2 a_2} \cdots e^{x_n a_n} \rangle \tag{A.7}$$

Note that this is a discrete version of Eq. (1.6). With Eq. (A.2) this implies

$$m(a_1 \cdot) = m^c(a_1 \cdot) * m \tag{A.8}$$

For a noncommutative state, m^c defined by Eq. (A.4) does not satisfy the cluster property, as seen from

$$\langle 12 \rangle^c = \langle 12 \rangle - \tfrac{1}{2}[\langle 1 \rangle \langle 2 \rangle + \langle 2 \rangle \langle 1 \rangle]$$

A2. K-Cumulants

For noncommutative states we use Eq. (A.7) or (A.8) to define an m^K in the sense of van Kampen,

$$m(a_1 \cdot) =: m^{\mathrm{K}}(a_1 \cdot) * m \tag{A.9}$$

or, equivalently, Eq. (A.7) with c replaced by K; from the latter, Eq. (3.7) follows. Now one cannot go back to Eq. (A.6), one only has the remnant of a logarithm. From Eq. (A.7) one immediately obtains Eq. (3.7), and with it one can also prove the cluster property, as in Section 3. Equation (A.9) together with Eq. (A.1) gives Eq. (1.7). Solving Eq. (A.9) for m^{K}, one gets

$$m^{\mathrm{K}}(a_1 \cdot) = m(a_1 \cdot) * m^{-1} = m(a_1 \cdot) * \sum_{l=0}^{\infty} (1 - m)_*^l \tag{A.10}$$

where 1 is the trivial state, which vanishes on all products of a_i. From this and Eq. (A.1) one immediately obtains Eq. (3.9). To express m in terms of m^{K}, we use Eq. (A.1) and iterate Eq. (A.9). This gives, with $\Lambda = \{1, \ldots, n\}$,

$$\langle a_\Lambda \rangle = \sum_{\substack{\Lambda_1 \cup \cdots \cup \Lambda_l = \Lambda \\ \min(\Lambda_j \cup \cdots \cup \Lambda_l) \in \Lambda_j}} \langle a_{\Lambda_1} \rangle^{\mathrm{K}} \cdots \langle a_{\Lambda_l} \rangle^{\mathrm{K}} \tag{A.11}$$

where $\langle a_\varnothing \rangle^{\mathrm{K}} := 0$.

In Eq. (A.10), a_1 is always on the left, but a_n is not always on the right. As a consequence of this asymmetry, m^{K} is not a Hermitean functional if m is not commutative. An alternative cumulant m^{R}, with a_n always on the right, can therefore be defined by

$$m^{\mathrm{R}}(a_1 \cdots a_n) := [m^{\mathrm{K}}((a_1 \cdots a_n)^*)]^* \tag{A.12}$$

which satisfies, by Eq. (A.9),

$$m(\cdot a_n) = m * m^{\mathrm{R}}(\cdot a_n) \tag{A.13}$$

These cumulants are adapted to the stochastic differential equation

$$\dot{U} = UV$$

If $\langle a_1 \cdots a_n \rangle^{\mathrm{K}} = 0$ for $n \neq 2$, i.e., in the K-Gaussian case, then $\langle a_A \rangle$ is a product of $\langle a_i a_j \rangle$; in particular,

$$\langle 1234 \rangle = \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle \tag{A.14}$$

Choosing $a_3 = a_2^*$ and $a_4 = a_1^*$, the lhs should be positive and *a fortiori* Hermitian. The first two terms on the rhs are indeed positive. The third, however, is $\langle a_1 a_1^* \rangle \langle a_2 a_2^* \rangle$, which is only Hermitian if the two factors commute. Hence, the K-Gaussian approximation to a state with noncommutative "second moments," i.e., retaining only second moments in Eq. (A.11), will not be positive and not a state. This is the underlying reason for the positivity problem in Section 3.

A3. W-Cumulants

We define a $\overset{*}{\sim}$-convolution by

$$S \overset{*}{\sim} T(a_I) := \sum_{I_1 \circ I_2 = I} S(a_{I_1})\, T(a_{I_2}) \tag{A.15}$$

where $I = \{1,..., n\}$; $I_1 = \{1,..., j\}$ and $I_2 = I \setminus I_1$ are adjoining and possibly empty. This defines an associative product. In a more general context a related product was introduced in Ref. 15. We define $m^{\mathrm{W}} \equiv \langle \cdot \rangle^{\mathrm{W}}$ by

$$m = 1 + m^{\mathrm{W}} \overset{*}{\sim} m \tag{A.16}$$

This is equivalent to Eq. (1.4), with $\langle a_\varnothing \rangle^{\mathrm{W}} = 0$. Iteration gives a Neumann series for m,

$$m = \sum_{l=0} (m^{\mathrm{W}})^l_{\overset{*}{\sim}} \tag{A.17}$$

or, equivalently,

$$\langle a_I \rangle = \sum_{I_1 \circ \cdots \circ I_l = I} \langle a_{I_1} \rangle^{\mathrm{W}} \cdots \langle a_{I_l} \rangle^{\mathrm{W}} \tag{A.18}$$

Solving Eq. (A.16) for m^{W} gives

$$m^{\mathrm{W}} = \sum_{l=0}^{\infty} (-1)^{l-1} (m-1)^l_{\sim} \tag{A.19}$$

from which Eq. (2.16) follows.

It is not difficult to show that the W-Gaussian approximation to a state, i.e., the linear functional built via Eq. (A.18) from the second moments alone, is always positive and thus a state. It can be shown that therefore no positivity problems arise with the W-Gaussian approximation to the integral equation in Section 2.

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